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On the iterates of a class of summation-type linear positive operators

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Abstract

This note is focused upon positive linear operators which preserve the quadratic test function. By using contraction principle, we study the convergence of their iterates.

Keywords: Linear positive operator; Popoviciu–Bohman–Korovkin theorem; Contraction principle; Modulus of continuity; Iterates

1. Introduction

Many of the linear methods of approximation are given by a sequence of linear positive operators (*LPOs*) designed as follows

$$(\Lambda_n f)(x) = \sum_{k=0}^{n} a_{n,k}(x) f(x_{n,k}), \quad f \in C([a,b]),\ x \in [c,d], \tag{1}$$

where every function $a_{n,k} \in C([c,d])$ is non-negative, $a = x_{n,0} < x_{n,1} < \cdots < x_{n,n} = b$ forms a mesh of nodes and $a \le c < d \le b$. As usual, we consider that the Banach space $C(K)$, $K \subset \mathbb{R}$ compact interval, is endowed with the norm $\|\cdot\|_K$ of the uniform convergence. In accordance with Popoviciu–Bohman–Korovkin theorem, if $(\|e_i - \Lambda_n e_i\|_{[c,d]})_n$ tends to zero for $i \in \{0,1,2\}$, then $(\Lambda_n f)_n$ converges uniformly to f for each f in $C([a,b])$. Here e_i stands for the monomial of i-degree.

King [1] constructed operators of Bernstein-type which reproduce the test functions e_0 and e_2. Starting from a similar class of operators having the degree of exactness zero, our aim is to study the convergence of the iterates and some approximation properties of our class as well.

2. The construction

As regards the operators defined by (1), we assume that the following identities

$$\sum_{k=0}^{n} a_{n,k}(x) = 1, \qquad \sum_{k=0}^{n} a_{n,k}(x) x_{n,k}^2 = x^2, \quad x \in [c,d],\ n \in \mathbb{N}, \tag{2}$$

are fulfilled.

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A generalization to the m-dimensional case will be read as follows. Let K_m be a compact and convex subset of the space $\mathbb{R}^m$. Volkov [2] proved that the functions: $\mathbf{1}$, $pr_1, \ldots, pr_m$, $\sum_{j=1}^m pr_j^2$, are test functions for $C(K_m)$. Here pr_j, $j = \overline{1, m}$, represent the canonical projections on K_m. Let $\Lambda_n : C(K_m) \to C(K_m)$ be such that

$$\Lambda_n \mathbf{1} = \mathbf{1} \quad \text{and} \quad \Lambda_n \left(\sum_{j=1}^m pr_j^2 \right) = \sum_{j=1}^m pr_j^2. \tag{3}$$

3. Results

By using the contraction principle we study the convergence of the iterates of the uni-dimensional operators Λ_n. We put $\Lambda_n^{m+1} = \Lambda_n \circ \Lambda_n^m$, $m \in \mathbb{N}$, and Λ_n^0 represents the identity operator of the space $C([a, b])$.

Theorem 3.1. *Let Λ_n, $n \in \mathbb{N}$, be defined by (1) and (2) such that $a = c < d = b$, $b \neq -a$ and $a_{n,0}(a) = a_{n,n}(b) = 1$. Set $u_n = \min_{x \in [a,b]} (a_{n,0}(x) + a_{n,n}(x))$. If $u_n > 0$, then the iterates sequence $(\Lambda_n^m)_{m \geq 1}$ verifies*

$$\lim_{m \to \infty} (\Lambda_n^m f)(x) = \frac{1}{b^2 - a^2} \left(f(a)b^2 - f(b)a^2 + (f(b) - f(a))x^2 \right), \tag{4}$$

uniformly with respect to x on $[a, b]$.

Considering the m-dimensional case of Λ_n described by (3), one has

$$\|\Lambda_n f - f\|_{K_m} \leq 2\omega(f; \sqrt{\mu_n}). \tag{5}$$

Here $\omega(f; \cdot)$ represents the modulus of continuity for the function f and

$$\mu_n := 2\|x\|^2 - 2 \sum_{i=1}^m x_i \Lambda_n(pr_i; x).$$

Proof. We define $X_{A,B} := \{f \in C([a, b]) \mid f(a) = A,\ f(b) = B\}$, $A \in \mathbb{R}$, $B \in \mathbb{R}$. Every $X_{A,B}$ is a closed subset of $C([a, b])$ and the system $X_{A,B}$, $(A, B) \in \mathbb{R} \times \mathbb{R}$, makes up a partition of this space. Since $a_{n,0}(a) = 1$, the first identity of (2) implies $a_{n,k}(a) = 0$, $k = \overline{1, n}$, consequently $(\Lambda_n f)(a) = f(a) = A$. Analogously, $(\Lambda_n f)(b) = f(b) = B$. These relations ensure that $X_{A,B}$ is an invariant subset of Λ_n for any $A \in \mathbb{R}$ and $B \in \mathbb{R}$.

Further on, we prove that the restriction of Λ_n at $X_{A,B}$ is a contraction for any $A \in \mathbb{R}$ and $B \in \mathbb{R}$. Indeed, if f and g belong to $X_{A,B}$ then, for each $a \leq x \leq b$, we can write

$$\begin{aligned} |(\Lambda_n f)(x) - (\Lambda_n g)(x)| &= \left| \sum_{k=1}^{n-1} a_{n,k}(x)(f - g)(x_{n,k}) \right| \leq \sum_{k=1}^{n-1} a_{n,k}(x) \|f - g\|_{[a,b]} \\ &= (1 - a_{n,0}(x) - a_{n,n}(x)) \|f - g\|_{[a,b]} \leq (1 - u_n) \|f - g\|_{[a,b]}, \end{aligned}$$

and consequently, $\|\Lambda_n f - \Lambda_n g\|_{[a,b]} \leq (1 - u_n) \|f - g\|_{[a,b]}$. On the other hand, the function

$$p_{A,B}^* = \frac{Ab^2 - Ba^2}{b^2 - a^2} e_0 + \frac{B - A}{b^2 - a^2} e_2$$

belongs to $X_{A,B}$. Since $\Lambda_n e_0 = e_0$, $\Lambda_n e_2 = e_2$, $p_{A,B}^*$ is a fixed point of Λ_n. For any $f \in C([a, b])$ one has $f \in X_{f(a), f(b)}$ and, by using the contraction principle, we get (4).

For the m-dimensional case, we can write

$$\Lambda_n(\| \cdot - x\|^2; x) = \Lambda_n \left(\sum_{i=1}^m (\cdot - x_i)^2; x \right) = 2\|x\|^2 - 2 \sum_{i=1}^m x_i \Lambda_n(pr_i; x),$$

where $\| \cdot \|$ stands for the Euclidean norm in $\mathbb{R}^m$. Taking into account a result established by Censor [3], Eq. (5), our relation (5) follows. □

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